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The complex Toda chains and the simple Lie algebras—solutions and large time asymptotics

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Abstract. The asymptotic regimes of the N-site complex Toda chain (CTC) with fixed ends related to the classical series of simple Lie algebras are classified. It is shown that the CTC models have much richer variety of asymptotic regimes than the real Toda chain (RTC). Besides asymptotically free propagation (the only possible regime for the RTC), CTC allows bound-state regimes, various intermediate regimes when one (or several) group(s) of particles form bound state(s), singular and degenerate solutions. These results can be used, for example, in describing the N-soliton train interactions of the nonlinear Schrödinger equation. Explicit expressions for the solutions in terms of minimal sets of scattering data are proposed for all classical series $\mathbf{B}_r$–$\mathbf{D}_r$.

1. Introduction

The Toda chain model [1–4]

$$\frac{\mathrm{d}^2 q_k}{\mathrm{d}t^2} = \exp(q_{k+1} - q_k) - \exp(q_k - q_{k-1}) \tag{1}$$

and its generalizations [5–15] is one of the paradigms of integrable nonlinear chains and lattices. It has been thoroughly studied for a number of initial and boundary conditions, such as:

- fixed ends boundary conditions, i.e. $q_0 = -q_{N+1} = \infty$; this will be the case we are interested in;
- infinite chain $-\infty < k < \infty$ with $\lim_{k\to-\infty} q_k = 0$ and $\lim_{k\to\infty} q_k =$ constant, or equivalently, $\lim_{k\to\pm\infty}(q_{k+1} - q_k) = 0$;
- quasi-periodic boundary conditions $q_{k+N} = q_k + c$, where $c =$ constant.

This model appeared first in describing the oscillations of a one-dimensional crystalline lattice [1]. Since then many other applications have become known, see for example [12].

The model (1) is directly related to the algebra $sl(N)$, where N is the number of sites of the chain. Most of the references cited above are devoted to the case when $q_k(t)$ are real-valued functions. That is why for definiteness we will call this model the real Toda chain (RTC). Other generalizations of the RTC are related to: (a) simple Lie algebras; (b) affine (or Kac–Moody) algebras; (c) two-dimensional generalizations.

Another possibility for generalizing the RTC, which as far as we know has not been investigated, is the complex Toda chain (CTC) which is described by (1) with complex-valued q_k and real time variable t. We see two main reasons for this.

(i) The generic solutions of the CTC are readily obtained from those of the RTC by simply making all dynamical parameters q_k complex. In fact, technically solving the RTC requires additional effort to ensure that the scattering data of the Lax matrix L are real valued.

(ii) The CTC has not been known to have physical applications.

Recently, however, it was discovered [16–18] that the CTC describes the N-soliton train interaction of the nonlinear Schrödinger (NLS) equation in the adiabatic approximation.

More specifically, here by N-soliton train we mean the solution to the NLS equation

$$\mathrm{i}u_T + \tfrac{1}{2}u_{xx} + |u|^2 u(x, T) = 0 \tag{2}$$

satisfying the initial condition

$$u(x, 0) = \sum_{k=1}^{N} u_k^{(1\mathrm{s})}(x, 0) \tag{3}$$

where

$$u_k^{(1\mathrm{s})}(x, T) = \frac{2\nu_k \exp(\mathrm{i}\phi_k(x, T))}{\cosh(2\nu_k(x - \xi_k(T)))} \tag{4a}$$

$$\phi_k(x, T) = 2\mu_k(x - \xi_k(T)) + \delta_k(T) \tag{4b}$$

$$\xi_k(x, T) = 2\mu_k T + \xi_{k,0} \tag{4c}$$

$$\delta_k(T) = 2(\mu_k^2 + \nu_k^2)T + \delta_{k,0}. \tag{4d}$$

Each term $u_k^{(1\mathrm{s})}(x, 0)$ is a one-soliton solution to the NLS equation with amplitude ν_k, velocity μ_k, initial centre-of-mass position $\xi_{k,0}$ and initial phase $\delta_{k,0}$. The adiabaticity condition means that the solitons are well separated $\xi_{k+1,0} - \xi_{k,0} \gg 1$ and have nearly equal initial amplitudes and velocities:

$$|\mu_{k,0} - \mu_{j,0}| \ll \mu_0 \qquad |\nu_{k,0} - \nu_{j,0}| \ll \nu_0 \qquad |\nu_{k+1,0} - \nu_{k,0}||\xi_{k+1,0} - \xi_{k,0}| \ll 1$$

where

$$\nu_0 = \frac{1}{N}\sum_{s=1}^{N} \nu_k \qquad \text{and} \qquad \mu_0 = \frac{1}{N}\sum_{s=1}^{N} \mu_k$$

are the average amplitude and velocity of the soliton train. Then the interaction can be viewed as a 'slow' evolution of the $4N$ soliton parameters. For definiteness we also assume that the initial positions $\xi_{k,0}$ of the solitons are ordered so that $\xi_{k+1,0} - \xi_{k,0} \simeq r_0$ are of the same order of magnitude; then only the nearest-neighbour interaction should be taken into account. More precisely, the results in [16–18] show that this evolution is provided by CTC where $t = 4\nu_0 T$ and the complex dynamical variables $q_k(t)$ in (1) are related to the soliton parameters of the kth soliton by

$$q_k(t) = -2\nu_0\xi_k(t) + k\ln 4\nu_0^2 + \mathrm{i}(2\mu_0\xi_k(t) - \delta_k(t) - \delta_0 t + k\pi). \tag{5}$$

Here, ξ_k and δ_k characterize the centre-of-mass position and the phase of the kth soliton in the train and $\delta_0 = 2(\mu_0^2 + \nu_0^2)$. Such soliton trains and their asymptotic behaviour appear to be important for the needs of soliton-based fibre optics communications.

One can also view the CTC as a model of N 'complex' particles on a line, each having two degrees of freedom. The kth particle is described by the complex function $q_k(t)$. The real and the imaginary parts of $q_k(t)$ and $\mathrm{d}q_k/\mathrm{d}t$ can be viewed as the dynamical variables of kth particle. Using (5) one can relate them to the parameters of the kth soliton in the N-soliton train (3).

Another reason for the present paper is in the fact that, along with the similarities between the solutions of the CTC and RTC, there are also important qualitative differences between the asymptotic properties of these solutions.

The purpose of the present paper is to derive analytically the large time asymptotics of the solutions to the CTC models

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} \equiv \sum_{k=1}^{r} \frac{\mathrm{d}^2 q_k}{\mathrm{d}t^2} H_k = \sum_{\alpha\in\pi_g} H_\alpha \,\mathrm{e}^{-(q,\alpha)} \tag{6}$$

related to the classical series of simple Lie algebras extending the results of [19, 20]. Assuming that the reader is familiar with the theory of simple Lie algebras [21, 22] we just recall the basic notions used here. By $\pi_g \equiv \{\alpha_1, \ldots, \alpha_r\}$ we mean the set of simple roots of the algebra $\mathfrak{g}$, $q(t)$ is a complex-valued function of t taking values in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; H_k forms a basis in $\mathfrak{h}$ dual to the orthonormal basis $\{e_k\}_{k=1}^{r}$ in the root space $\mathbb{E}^r$ and r is the rank of $\mathfrak{g}$; $H_\alpha = \sum_{k=1}^{r} (\alpha, e_k) H_k$. For more details about the structure of the simple Lie algebras, see, for example, [21]. Equation (1) is a particular case of (6) for the $\mathbf{A}_r$ series, i.e. for $\mathfrak{g} \simeq sl(r+1)$. These results, and especially those for the series $\mathbf{A}_r$, can be used as a tool for deriving the asymptotic behaviour of the N-soliton trains of the NLS equation from the initial set of soliton parameters [16–18].

We also specify the minimal sets of scattering data $\mathcal{T}_g$ for L which determine uniquely both L and the solutions of (6) and obtain explicit expressions for the solutions of (6) in terms of $\mathcal{T}_g$ which are compatible with the ones in [6, 7].

2. Comparison between RTC and CTC

Since the paper of Moser [4] on the finite non-periodic real Toda lattice, several methods have been proposed for the solution of the RTC and its generalizations for various choices of the initial and boundary conditions, see [1–12, 23–27] and the numerous references therein. The RTC model was also extended to indefinite metric spaces [26–28] and was shown to possess singular solutions 'blowing up' for finite values of t. These models can be viewed as special cases of the CTC in which part of the a_k defined later are real while the rest are purely imaginary. A series of papers [13–15] is devoted to the thorough study of the singularities of the solutions of the CTC (their order, positions and structure) in the most general case when not only the dynamical parameters q_k, but also the time t become complex.

As we have already mentioned, a number of properties of the CTC are obtained trivially from the corresponding ones of the RTC by assuming that the corresponding dynamical variables are complex. We list the four most important properties.

(S1) The Lax representation. There are several equivalent formulations of the Lax representation for (6). In the following we will use the 'symmetric' one:

$$L(t) = \sum_{k=1}^{r} (b_k H_k + a_k (E_{\alpha_k} + E_{-\alpha_k})) \tag{7a}$$

$$M(t) = \sum_{k=1}^{r} a_k (E_{\alpha_k} - E_{-\alpha_k}) \tag{7b}$$

where $a_k = \frac{1}{2}\exp(-(q,\alpha_k)/2)$ and $b_k = -\frac{1}{2}\,\mathrm{d}q_k/\mathrm{d}t$; for $\mathfrak{g} \simeq sl(N)$ we have $a_k = \frac{1}{2}\exp((q_{k+1} - q_k)/2)$. It is well known that to each root $\alpha \in \Delta_g \subset \mathbb{E}^r$ one can put into correspondence the element $H_\alpha \in \mathfrak{h}$. Analogously, to $q(t) = \mathrm{Re}\, q(t) + \mathrm{i}\,\mathrm{Im}\, q(t)$ there corresponds the vector $\boldsymbol{q}(t) = \mathrm{Re}\,\boldsymbol{q}(t) + \mathrm{i}\,\mathrm{Im}\,\boldsymbol{q}(t)$, whose real and imaginary parts are vectors in $\mathbb{E}^r$.

(S2) The integrals of motion in involution are provided by the eigenvalues, ζ_k, of L.

(S3) The solutions of both the CTC and the RTC are determined by the scattering data for $L_0 \equiv L(0)$. When the spectrum of L_0 is non-degenerate, i.e. $\zeta_k \neq \zeta_j$ for $k \neq j$, then this scattering data consists of

$$\mathcal{T} \equiv \{\zeta_1, \ldots, \zeta_N, r_1, \ldots, r_N\} \tag{8}$$

where r_k are the first components of the corresponding eigenvectors $v^{(k)}$ of L_0 in the typical representation $R(\omega_1)$ of $\mathfrak{g}$, $N = \dim R(\omega_1)$,

$$L_0 v^{(k)} = \zeta_k v^{(k)} \tag{9}$$

or if we introduce the matrix V with $V_{sk} = v_s^{(k)}$ we have

$$L_0 V = V Z \qquad Z = \mathrm{diag}(\zeta_1, \ldots, \zeta_N). \tag{10}$$

The quantities $r_k = v_1^{(k)} = V_{1k}$ are determined (up to an overall sign) by the normalization conditions:

$$\sum_{s=1}^{N} (V_{sk})^2 = (v^{(k)}, v^{(k)}) = 1 \qquad k = 1, \ldots, N \tag{11}$$

see [4, 11, 12]; then $V^{\mathrm{T}} = V^{-1}$.

(S4) Lastly, the eigenvalues of L_0 uniquely determine the asymptotic behaviour of the solutions; these eigenvalues can be calculated directly from the initial conditions. We will extensively use this fact for the description of the different types of asymptotic behaviour.

However, there are important differences between the RTC and CTC, in particular the asymptotic behaviour of their solutions. Indeed, for the RTC, one has [4, 12] that both the eigenvalues, ζ_k, and the coefficients, r_k, are always real valued. Moreover, one can prove that $\zeta_k \neq \zeta_j$ for $k \neq j$, i.e. no two eigenvalues can be exactly the same. As a direct consequence of this, it follows that the only possible asymptotic behaviour in the RTC is an asymptotically separating, free motion of the particles (solitons).

This situation is different for the CTC. Now the eigenvalues $\zeta_k = \kappa_k + \mathrm{i}\eta_k$, as well as the coefficients r_k, become complex. Furthermore, the argument of Moser cannot be applied so one can have multiple eigenvalues. The collection of eigenvalues, ζ_k, still determines the asymptotic behaviour of the solutions. In particular, it is κ_k that determines the asymptotic velocity of the kth particle (soliton). For simplicity, we assume $\zeta_k \neq \zeta_j$ for $k \neq j$. However, this condition does not necessarily mean that $\kappa_k \neq \kappa_j$. We also assume that the κ_ks are ordered as

$$\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_N. \tag{12}$$

Once this is done, then for the corresponding set of N particles (train of N solitons), there are three possible general configurations.

(D1) $\kappa_k \neq \kappa_j$ for $k \neq j$. Since the asymptotic velocities are all different, one has the asymptotically separating, free particles (solitons).

(D2) $\kappa_1 = \kappa_2 = \cdots = \kappa_N$. In this case, all N particles (solitons) will move with the same mean asymptotic velocity, and therefore will form a ‘bound state’. The key question now will be the nature of the internal motions in such a bound state.

(D3) One may also have a variety of intermediate situations when only one group (or several groups) of particles move with the same mean asymptotic velocity; then they would form one (or several) bound state(s) and the rest of the particles will have free asymptotic motion.

Obviously, the cases (D2) and (D3) have no analogues in the RTC and physically are qualitatively different from (D1). The same is also true for the special degenerate cases, where two or more of the ζ_ks may become equal and for the singular solutions. These cases will be considered briefly in the following.

3. Solutions of the CTC

The solutions for the CTC can be obtained formally from the well known ones for RTC by inserting the corresponding complex parameters. We note first the solution of the RTC for $\mathfrak{g} \simeq sl(N)$, see [4, 12, 29] and references therein. Here we fix up the mass centre at the origin by

$$\sum_{k=1}^{N} q_k(t) = 0. \tag{13}$$

The velocity of the centre of masses is given by $\operatorname{tr} L_0 = \sum_{k=1}^{N} \zeta_k = 0$, due to $L \in sl(N)$. Then the solution has the form

$$q_k(t) = q_1(0) + \ln \frac{A_k}{A_{k-1}} \tag{14}$$

where $A_0 \equiv 1$,

$$A_1(t) = \sum_{k=1}^{N} r_k^2 \, \mathrm{e}^{-2\zeta_k t} \tag{15}$$

$$A_k(t) = \sum_{1 \leqslant l_1 < l_2 < \cdots < l_k \leqslant N} (r_{l_1} r_{l_2} \ldots r_{l_k})^2 W^2(l_1, l_2, \ldots, l_k) \, \mathrm{e}^{-2(\zeta_{l_1} + \cdots + \zeta_{l_k})t} \tag{16}$$

and

$$A_N = W^2(1, 2, \ldots, N) \prod_{k=1}^{N} r_k^2 = \exp(-N q_1(0)). \tag{17}$$

By $W(l_1, \ldots, l_k)$ we denote the Vandermonde determinant:

$$W(l_1, \ldots, l_k) = \prod_{\substack{s > p \\ s, p \in \{l_1, \ldots, l_k\}}} (2\zeta_s - 2\zeta_p). \tag{18}$$

The solutions of (1) for real-valued $q(t)$ are well known in several different formulations. We note here the formula, which effectively is contained in [6],

$$(\boldsymbol{q}(t), \omega_k) - (\boldsymbol{q}(0), \omega_k) = \ln\langle \omega_k | \exp(-2L_0 t) | \omega_k \rangle \tag{19}$$

where ω_k are the fundamental weights of $\mathfrak{g}$. The fact that the large time asymptotics of $q_k(t)$ for the $\mathfrak{g} \simeq sl(N)$ RTC have the form

$$\lim_{t \to \pm\infty} (q_k(t) - v_k^{\pm} t) = \beta_k^{\pm} \tag{20}$$

where the asymptotic velocities $v_k^+ = -2\zeta_k$ and $v_k^- = -2\zeta_{N+1-k}$, has been derived by Moser [4]. He also evaluated the differences $\beta_k^+ - \beta_{N-k+1}^-$ which characterize the interaction in the RTC model.

The minimal set of scattering data for $\mathfrak{g} \simeq sl(N)$ is obtained from (8) by imposing on $\mathcal{T}$ the restrictions

$$\sum_{k=1}^{N} \zeta_k = 0 \tag{21a}$$

and

$$\sum_{k=1}^{N} r_k^2 = 1. \tag{21b}$$

Note that $\exp(-q_1(0))$ is expressed through $\mathcal{T}$ by (17).

For the RTC related to the other classical series of Lie algebras it is known [19, 20] that

$$\lim_{t\to\pm\infty} (q(t) - v^{\pm}t) = \beta^{\pm} \tag{22}$$

$v^{\pm} \in \mathfrak{h}$, $\beta^{\pm} \in \mathfrak{h}$ and $v^{+} = w_0(v^{-})$, where w_0 is the element of the Weyl group, which maps the highest weight of each irreducible representation of $\mathfrak{g}$ into the corresponding lowest weight. The action of w_0 on the simple roots is well known [21, 22]:

$$w_0(\alpha_k) = -\alpha_{\tilde{k}} \tag{23}$$

where $\tilde{k} = r - k + 1$ for $\mathbf{A}_r$; $\tilde{k} = k$, $k = 1, \dots, r$ for $\mathbf{B}_r$, $\mathbf{C}_r$ and, when r is even, also for $\mathbf{D}_r$. When $\mathfrak{g} \simeq \mathbf{D}_r$ and r is odd $\tilde{k} = k$ for $k \leqslant r - 2$, and $w_0(\alpha_{r-1}) = -\alpha_r$, $w_0(\alpha_r) = -\alpha_{r-1}$.

What we will do in the following is to: (i) specify how minimal sets of scattering data $\mathcal{T}_g$ can be extracted from (8); (ii) find explicit expressions for $\beta_k^{\pm} \equiv (\beta^{\pm}, e_k)$ for each of the classical Lie algebras in terms of $\mathcal{T}_g$.

As in the $sl(N)$-case, ζ_k and r_k are the eigenvalues and the first components of the eigenvectors of L_0 in the typical representation, namely

$$L_0 V = V \sum_{k=1}^{r} \zeta_k H_k. \tag{24}$$

The requirement that L_0 (and as consequence, L) belongs to one of the algebras in the $\mathbf{B}_r$ or $\mathbf{C}_r$ series imposes on q_k the following natural restrictions,

$$q_k = -q_{N-k+1} \tag{25a}$$

which leads to

$$a_k = a_{N-k} \tag{25b}$$

$$b_k = -b_{N+1-k}. \tag{25c}$$

Thus, the solutions for $\mathfrak{g} \simeq \mathbf{B}_r$ and $\mathbf{C}_r$ can formally be obtained from those for $sl(N)$ (14)–(17) by imposing on them the involutions (25a)–(25c). So we have to find out what are the restrictions on $\mathcal{T}$ imposed by (25a)–(25c); this will provide us with the corresponding minimal set of scattering data $\mathcal{T}_g$, which must obviously contain only $2r$ parameters. It is easy to find that $\zeta_{\bar{k}} = -\zeta_k$, where $\bar{k} = N + 1 - k$, so only r of them are independent. It is not so trivial to derive the corresponding relations which reduce the number of the coefficients r_k. Our analysis shows that

$$r_k r_{\bar{k}} = \exp(-q_1(0)) w_k \qquad k = 1, \dots, r. \tag{26}$$

Now we provide the explicit formulae for w_k for each of the classical series $\mathbf{B}_r$, $\mathbf{C}_r$ and $\mathbf{D}_r$.

$\mathbf{B}_r$*-series*. $N = 2r + 1$. Note that in this case $\zeta_{r+1} = 0$ and, in addition to (26),

$$r_{r+1}^2 = \exp(-q_1(0)) \frac{1}{2^{2r}(\zeta_1\zeta_2\dots\zeta_r)^2} \tag{27}$$

and the expression for w_k is provided by

$$w_k = \frac{1}{8\zeta_k^2} \prod_{s=1}^{k-1} \frac{1}{4\zeta_s^2 - 4\zeta_k^2} \prod_{s=k+1}^{r} \frac{1}{4\zeta_k^2 - 4\zeta_s^2}. \tag{28}$$

Inserting (26)–(28) into (21b) we obtain a quadratic equation for $\exp(-q_1(0))$, so it can be expressed in terms of $\mathcal{T}_g$.

C$_r$*-series.* $N = 2r$. Here

$$w_k = -\frac{1}{4\zeta_k} \prod_{s=1}^{k-1} \frac{1}{4\zeta_s^2 - 4\zeta_k^2} \prod_{s=k+1}^{r} \frac{1}{4\zeta_k^2 - 4\zeta_s^2}. \tag{29}$$

As for the **B**$_r$-series, $\exp(-q_1(0))$ is determined from (21b).

D$_r$*-series.* $N = 2r$. Here

$$w_k = \prod_{s=1}^{k-1} \frac{1}{4\zeta_s^2 - 4\zeta_k^2} \prod_{s=k+1}^{r} \frac{1}{4\zeta_k^2 - 4\zeta_s^2}. \tag{30}$$

Again $\exp(-q_1(0))$ is determined from (21b) and (26). The derivation of the solution for this series requires some additional effort. The main problem here is to find explicit parametrization for the right-hand sides of (19) for $k = r - 1$ and r in terms of r_k, which characterize the matrix V in (24) in the typical representation. Skipping the details we just present the result, namely that q_k with $k = 1, \ldots, r - 1$ are again given by (14)–(16) where $\zeta_k = -\zeta_{\bar{k}}$ and r_k are restricted by (26) and (30). For $k = r$ the solution for $q_r(t)$ is provided by (14) with

$$A_r(t) = \sum_{1 \leqslant l_1 < l_2 < \cdots < l_r \leqslant N} (r_{l_1} r_{l_2} \ldots r_{l_r})^2 W^2(l_1, l_2, \ldots, l_r) f_{l_1,\ldots,l_r}^2 \, \mathrm{e}^{-2(\zeta_{l_1} + \cdots + \zeta_{l_r})t} \tag{31}$$

and

$$f_{l_1,\ldots,l_r} = \frac{1}{2}\left(1 + \frac{\zeta_1 \zeta_2 \ldots \zeta_r}{\zeta_{l_1} \zeta_{l_2} \ldots \zeta_{l_r}}\right). \tag{32}$$

The proof of these formulae is based on detailed analysis of the properties of the fundamental representations $R(\omega_k)$ of $\mathfrak{g}$ and of their tensor products.

We note that these formulae are valid both for RTC and CTC. In the latter case one should be careful to avoid the subset of singular solutions, when one or more of the functions $A_k(t)$ may develop zeros for finite values of t, see [13, 14, 26–28] and the discussion in section 5.

4. Large time asymptotics

Let us now express large time asymptotics of $q_k(t)$ in terms of the minimal set of scattering data $\mathcal{T}_g$ and analyse the different types of asymptotic regimes.

As we mentioned above, we view the CTC as a model, describing non-trivial scattering of N 'complex particles' so that $\operatorname{Re} q_k(t)$ and $\operatorname{Im} q_k(t)$ correspond to their coordinates and 'phases'.

(D1) *Asymptotically free states* $\kappa_k \neq \kappa_j$ for $k \neq j$. This regime is specified by imposing on $\zeta_k = \kappa_k + \mathrm{i}\eta_k$, $k = 1, \ldots, N$ the so-called sorting condition:

$$\kappa_1 < \kappa_2 < \cdots < \kappa_N. \tag{33}$$

Now we have to express $\beta_k^{\pm}$ in terms of $\mathcal{T}_g$. Skipping the details we list the results for the classical series of simple Lie algebras **A**$_r$–**D**$_r$.

$\mathbf{A}_r$*-series*. $N = r + 1$. The corresponding $\mathcal{T}_g$ is formed from $\{\zeta_k, r_k\}_{k=1}^{N}$ by imposing $\sum_{k=1}^{N} \zeta_k = 0$ and the normalization condition (21*b*). Then

$$\beta_k^+ = q_1(0) + \ln r_k^2 + \ln \prod_{s=1}^{k-1} (2\zeta_s - 2\zeta_k)^2 \qquad (34a)$$

$$\beta_k^- = q_1(0) + \ln r_{\bar{k}}^2 + \ln \prod_{s=\bar{k}+1}^{N} (2\zeta_s - 2\zeta_{\bar{k}})^2 \qquad (34b)$$

where $\bar{k} = N + 1 - k$.

Now it is easy to calculate the shift of the relative position which is the effect of the particle interaction. Naturally these shifts are also complex valued. If q_k correspond to NLS solitons then $\beta_k^+ - \beta_k^-$ will describe the shifts of both the relative positions and phases of the solitons, see formula (5). Note that in the class of regular solutions the particles do not collide, i.e. their trajectories do not intersect. Since we have ordered the particles by their velocities assuming $\kappa_1 < \kappa_2 < \cdots < \kappa_N$, so for $t \to \infty$ the kth particle will move with velocity $-2\kappa_k$. For $t \to -\infty$, however, we find that due to the interaction now the kth particle moves with velocity $-2\kappa_{N-k+1} = -2\kappa_{\bar{k}}$. This is the so-called 'sorting property' characteristic for the RTC. If we identify the kth particle (soliton) by its velocity then its shift of position will be given by the real part of the relation

$$\beta_{\bar{k}}^- - \beta_k^+ = \sum_{j \neq k} \epsilon_{jk} \ln(2\zeta_j - 2\zeta_k)^2 \qquad (35)$$

where $\epsilon_{jk} = 1$ for $j > k$ and $\epsilon_{jk} = -1$ for $j < k$. The imaginary part of (35) will provide the shift in the phase of the corresponding particle (soliton). Equation (35) is a natural generalization of the corresponding result of Moser [4] for the RTC.

The asymptotics for the $\mathbf{B}_r$, $\mathbf{C}_r$ and $\mathbf{D}_r$ series have the form (22)

$$\lim_{t \to \pm\infty} (q_k(t) \pm 2\zeta_k t) = \beta_k^{\pm} \qquad (36)$$

with

$$\beta_k^+ = q_1(0) + \ln r_k^2 + \ln \prod_{s=1}^{k-1} (2\zeta_s - 2\zeta_k)^2 \qquad (37a)$$

$$\beta_k^- = -q_1(0) + \ln \frac{w_k^2}{r_k^2} + \ln \prod_{s=1}^{k-1} (2\zeta_s - 2\zeta_k)^2 \qquad (37b)$$

for $k = 1, \ldots, r$.

The only exception is for the series $\mathbf{D}_r$ in the case of odd r, $k = r$ and $t \to -\infty$ which reads

$$\lim_{t \to -\infty} (q_r(t) + 2\zeta_r t) = \beta_r^- \qquad (38)$$

where

$$\beta_r^- = q_1(0) + \ln r_r^2 + \ln \prod_{s=1}^{r-1} (2\zeta_r - 2\zeta_{\bar{s}})^2. \qquad (39)$$

(D2) *Bound states.* $\kappa_1 = \kappa_2 = \cdots = \kappa_N = 0$. Now all N particles will move with the same mean asymptotic velocity for both $t \to \infty$ and $t \to -\infty$; by Galilean transformation this velocity can always be made zero. The individual velocities of the particles oscillate around the common mean value. In other words, we find that all N particles generically do not separate but form a bound state with $2N$ degrees of freedom. The explicit solutions for q_k now do not simplify even for $t \to \pm\infty$. Nevertheless, two features are worth noting.

The solutions will be periodic functions of t provided the ratios $(\eta_k - \eta_m)/(\eta_k - \eta_j)$ are rational numbers for all k, m and j. In some cases they can also become singular, see the next section.

Important for possible physical applications is the so-called quasi-equidistant regime in which the distances between the neighbouring particles, i.e. $\mathrm{Re}(q_{k+1}(t) - q_k(t))$ oscillate with very small amplitude; with rather good accuracy they could be considered constant. The fact that such regimes are possible and describe adequately the behaviour of certain NLS soliton trains is shown in [30].

(D3) *Mixed regimes.* As we mentioned above, there are a number of intermediate cases. Here we start with the case when $m+1$ out of the N particles form a bound state, i.e.

$$\kappa_1 < \cdots < \kappa_k = \cdots = \kappa_{k+m} < \kappa_{k+m+1} < \cdots < \kappa_N \tag{40}$$

and $\eta_i \neq \eta_j$ for $i \neq j \in \{k, k+1, \ldots, k+m\}$. Skipping the details, we present the results for the case $\mathfrak{g} \simeq sl(N)$ and $m = 1$, i.e. only two of the particles form a bound state. Particles with numbers different from k, $k+1$ are free, and for kth and $k+1$th we have

$$q_{k+a}(t) = q_1(0) + u_k t + \beta^+_{k+a}(t) + \mathrm{O}(\mathrm{e}^{-2K_k t}) \tag{41}$$

for $t \to \infty$ and

$$q_{N-k+a}(t) = q_1(0) + u_k t + \beta^-_{k+a}(t) + \mathrm{O}(\mathrm{e}^{2K_k t}) \tag{42}$$

for $t \to -\infty$. Here $a = 0, 1$, $u_k = -(\zeta_k + \zeta_{k+1})$,

$$K_k = \min(\kappa_k - \kappa_{k-1}, \kappa_{k+2} - \kappa_{k+1})$$

and

$$\beta^\pm_{k+a} = \tilde{\beta}^\pm_k + (-1)^a B^\pm_k(t) + a \ln(2\zeta_k - 2\zeta_{k+1})^2 \tag{43}$$

$$B^\pm_k(t) = \ln(2\cos((\eta_k - \eta_{k+1})t - \mathrm{i}\gamma^\pm_k)) \tag{44}$$

$$\tilde{\beta}^\pm_k = \ln\left(r_k r_{k+1} \prod_{s,k}^{\pm} (2\zeta_s - 2\zeta_k)(2\zeta_s - 2\zeta_{k+1}) \right) \tag{45}$$

$$\gamma^\pm_k = \ln\left(\frac{r_{k+1}}{r_k} \prod_{s,k}^{\pm} \frac{\zeta_s - \zeta_{k+1}}{\zeta_s - \zeta_k} \right) \tag{46}$$

where $\prod^+_{s,k} \equiv \prod_{s=1}^{k-1}$ and $\prod^-_{s,k} \equiv \prod_{s=k+2}^{N}$.

From (43) one can find the shifts due to the interaction.

5. On the singular and degenerate solutions of CTC

It is only natural that some of the CTC solutions do not enjoy all the good properties of the RTC ones. We have mentioned already two of them:

(P1) the elements of $\mathcal{T}_g$ for the RTC are real valued;

(P2) The eigenvalues ζ_k are pairwise different, i.e. $\zeta_k \neq \zeta_j$ for $k \neq j$.

An immediate consequence of (P1) and (P2) is the fact that the solutions $q_k(t)$ of the RTC are regular functions for all finite values of t.

Generalizing to the CTC, we loose both properties (P1) and (P2). As a result, besides the regular solutions, we also obtain singular and degenerate solutions. Obviously, all solutions leading to the D1 regime are regular. Even if we assume that property (P2) holds, we may still have singular solutions.

Indeed, from equations (44)–(46) we see that in the 'oscillating part' of the motion $B_k^{\pm}(t)$ is periodic. If, in addition, the parameters in $\mathcal{T}_g$ are such that $\operatorname{Re}\gamma_k^{\pm} = 0$ in (46) then $B_k(t)$ will develop singularities for finite values of t. The same holds true for the functions $q_k(t)$: there exist submanifolds of $\mathcal{T}_g$ for which $q_k(t)$ become singular for finite values of t.

As we mentioned above, this fact is compatible with the results of [13–15] who have proposed a method for description of the varieties of singular points of the CTC. Some of their results have been extended for the generalized RTC [27] and also for an RTC on spaces with indefinite metric [26] and (or) with purely imaginary interaction constant [28]. Such an RTC can be viewed as particular case of the CTC when one or several of the functions $a_k(t)$ are purely imaginary, while the others remain real.

Let us now examine the degeneration, i.e. the case when the property (P2) is violated and some of the eigenvalues of L_0 become equal. In [4, 31] it is proved that the spectrum of L_0 is simple for a real $sl(N)$-Toda chain. We state the following generalization.

Lemma 1. Let us consider L_0 for the complex Toda models related to the classical series of simple Lie algebras $A_r,\ B_r,\ C_r,\ D_r$. If L_0 does not contain Jordan cells, then its spectrum is simple.

Therefore, the degeneration can take place only for CTC models and only provided Jordan cells in the diagonalization of L_0 are present. Let us have $\mathfrak{g} \simeq sl(N)$ and let L_0 have a 2×2 Jordan cell, $\zeta_1 = \zeta_2 = \zeta$. Then L_0 has an eigenvector $v^{(1)}$ and an adjoint eigenvector $v^{(2)}$:

$$L_0 v^{(1)} = \zeta v^{(1)} \tag{47a}$$

$$L_0 v^{(2)} = \zeta v^{(2)} + v^{(1)} \tag{47b}$$

where $v^{(2)}$ can be expressed as a linear combination of $v^{(1)}(\zeta)$ and its first derivative with respect to ζ. The corresponding $A_k(t)$ besides the standard exponential terms will also contain terms of the form $t\,\mathrm{e}^{-2\zeta t}$. More generally, if the degeneracy is of higher order, i.e. $\zeta_1 = \cdots = \zeta_m$ then we will need linear combinations of $v^{(1)}(\zeta)$ and its derivatives with respect to ζ of order $1, \ldots, m-1$ and A_k will contain terms of the form $t^p\,\mathrm{e}^{-2n\zeta t}$ with $p = 1, \ldots, m-1$ and $n = 1, \ldots, \min(k, m)$, see also [32].

In particular, if we have complete degeneracy (i.e. all ζ_k are equal and equal to zero) the solution of the $sl(N)$-CTC is expressed through A_k, which are polynomials of degree $k(N-k)$. Their coefficients depend on $N-1$ constants $f_k,\ k = 1, \ldots, N-1$. For example, for $N = 3$ and $\zeta_1 = \zeta_2 = \zeta_3 = 0$ we get

$$A_1(t) = -\tfrac{1}{2}t^2 + f_1 t + f_2 \tag{48}$$

$$A_2(t) = -\tfrac{1}{2}t^2 + f_1 t - f_1^2 - f_2 \tag{49}$$

and $A_3 = 1$. Obviously, these solutions will be regular if A_k have complex roots and will develop singularities if one (or more) of their roots are real.

6. Conclusions

We have shown that the N-site CTC with fixed ends has a richer variety of asymptotical regimes than the RTC. We have also evaluated the large time asymptotics of $q_k(t)$ which may be used in the studies of N-soliton train interactions.

In particular, we showed that the CTC allows solutions in which the 'complex' particles form regular bound states. This could be important for the applications to soliton interactions, see for example [30]. Then the problem to determine the initial soliton parameters (i.e. the values of a_k and b_k) is reduced to the analysis of the characteristic equation for L_0,

$$\det(L_0 - \zeta) = \sum_{k=0}^{N} p_k \zeta^k \tag{50}$$

$p_N = 1,\ p_{N-1} = 0$, and to the requirement that the eigenvalues of L_0 be purely imaginary. This is an algebraic problem which often can be solved analytically. It allows one to determine the set of initial soliton parameters in such a way that the solitons will not only form a stable bound state, but also will propagate quasi-equidistantly. This type of propagation is of importance for soliton-based fibre optics communications [30].

We have also studied the asymptotic regimes for the CTC related to the other simple Lie algebras. We have proposed explicit solutions to these models in terms of the minimal sets of scattering data $\mathcal{T}_g$. The degenerate and singular solutions of the CTC which have no counterparts in RTC are also briefly analysed and are compatible with the earlier known results of [13–15, 26–28].

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